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Asymptotic-Diffusion Analysis of the Retrial Queueing System $M^{(2)}|M^{(2)}|1$ with Priority Customers for a Non-Priority Component

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Abstract—In this paper, we consider retrial queueing system. Two classes of customers come to the system according Poisson arrival processes. The first flow is a flow of priority customers, the second flow is a flow of non-priority customers. Service times have exponential distributions. If a priority customer finds the server occupying by the customer of the same class, it goes to an orbit (orbit for priority customers) and makes a repeated attempt after a random delay. Inter-retrial times have exponential distributions. If an arrival priority customer finds the non-priority customer on the server, it can interrupt its service and starts servicing itself. The preempted customers moves into the orbit for non-priority customers. If a non-priority customer finds the server occupying, it goes to an orbit (orbit for non-priority customers). The customers from the orbit behave the same way. Customers are submitted to the system after successful completion of service. We propose an asymptotic-diffusion analysis of the system. Probability distribution of the number of customers in a non-priority orbit and in a priority orbit are obtained.

Keywords: queueing system, retrial queueing system, orbit, asymptotic-diffusion analysis, diffusion process, probability distribution

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1. INTRODUCTION

The formation of queues in bank offices, hospitals, shopping centers and other facilities is a common problem that modern people face in every day life. Sometimes, unable to wait in queue, people leave and attempt to get service later, when the load is reduced. This phenomenon has led to special systems in the queue theory, called retrial queueing systems (RQ-systems). These systems are characterized by the customer (call) finding the server occupied, moves to a virtual orbit where it remains for some random time, then attempts to be served again. A significant effort has been done by a number of scientists involved in RQ-systems research. An overview of this topic can be found in the following sources [1–5].

But in practice, traditional RQ-system models are difficult to apply widely in real-life conditions due to their lack of versatility. Almost every field of life is affected by the prioritization of one customer over another. In service systems such as airlines, first class customers are prioritized over economy class. In telecommunication system, voice packs have higher priority than other data packs like email etc. In [6] various RQ-systems with priority calls and many other system parameters like customer loss, feedback etc. are analyzed.

Recently, mobile traffic has increased rapidly, resulting in a shortage of wireless spectrum. Cognitive radio networks are promising technologies to solve the spectrum scarcity problem. In cognitive radio networks, there are priority and non-priority users. Primary users (priority users) provide some spectrum bands to secondary users (non-priority users). Secondary users can cognitively utilize these bands when they are not used by the primary users, but when the primary user has arrived for service, the current secondary user must leave the server and attempt to occupy the channel again later. From a mathematical point of view, the study of RQ-systems with two types of customers is significantly more difficult than with a single type of customer. A number of works are devoted to the study of systems with two flows. In [7], a priority system with urgent requests and heterogeneous service is studied. In [8], the authors considered a system in which a prioritized request, having found a server occupied, either preempts a request on the server or queues up with some probability. In [8], the authors take into account the possibility of server breakdown. In [9–11] the functioning metrics of RQ-systems of different configurations with two flows (priority and non-priority) and a queue for priority flows are also found.

Notably, in the works described above, the priority customers, having found the server occupied, enter the queue. In this paper, we propose a system with repeated calls, in which priority customers, as well as non-priority ones, go into orbit, i.e., we need to consider a system with two orbits. In [12], the authors considered a tandem RQ-system with two orbits. The peculiarity of the system studied in this paper is that prioritized customers can displace the serviced non-prioritized customers.

2. MATHEMATICAL MODEL AND PROBLEM STATEMENT

Consider a system with repeated calls (retrial queueing system, RQ-system), Fig. 1.

Two simple flows with parameters λ_1 and λ_2 are received. The first flow is the flow of priority customers and the second is the flow of non-priority customers. When at the moment of arrival a customer finds the server free, it starts to be serviced during the time distributed according to the exponential law with parameters μ_1 and μ_2 or the priority and non-priority flows calls, respectively. After successful completion of the service, the customer leaves the system. When a priority flow customer finds the server occupied at the time of arrival, then:

- if a priority customer was serviced at the server, the incoming one moves to the orbit where it performs a random delay having an exponential distribution with parameter σ_1. After a random delay, it reaccesses the server with a second attempt to capture again;
- if a non-priority customer was serviced at the server, the incoming customer displaces the serviced one and starts to be serviced itself, and the preempted one moves to the orbit for non-priority customers, where it performs a random delay exponentially distributed with the parameter σ_2, after which it contacts the device with a repeated attempt to capture the server.

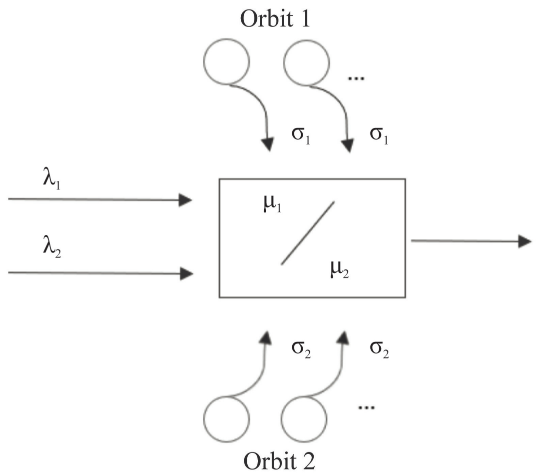

Fig. 1. Mathematical model of the RQ-system $M^{(2)}|M^{(2)}|1$.

If it's free it starts service, if it's occupied it goes back instantly. The discipline for addressing priority calls from orbit to the server is the same as for addressing priority calls newly arrived in the system. Denoting: $i_1(t)$ — number of customers in the first orbit, $i_2(t)$ — number of customers in the second orbit. The server state is determined by the value $k(t)$: $k(t) = 0$, if the server is free for service; $k(t) = 1$, if the server is occupied with servicing a priority customer; $k(t) = 2$, if the server is occupied with servicing a non-priority customer. Then the probability that at time t the device is in state k, in the first orbit the number of customers is i_1, in the second orbit the number of customers is i_2, denote as

$$P\{k(t) = k, i_1(t) = i_1, i_2(t) = i_2\} = P_k(i_1, i_2, t). \tag{1}$$

For the distribution (1), using the formula of full probability, we make a system of Kolmogorov differential equations:

$$\begin{aligned}
\frac{\partial P_0(i_1,i_2,t)}{\partial t} &= -(\lambda_1 + \lambda_2 + i_1\sigma_1 + i_2\sigma_2)P_0(i_1,i_2,t) + \mu_1 P_1(i_1,i_2,t) + \mu_2 P_2(i_1,i_2,t),\\
\frac{\partial P_1(i_1,i_2,t)}{\partial t} &= -(\lambda_1 + \lambda_2 + \mu_1)P_1(i_1,i_2,t) + \lambda_1 P_1(i_1 - 1, i_2, t)\\
&+ \lambda_2 P_1(i_1, i_2 - 1, t) + \lambda_1 P_0(i_1,i_2,t) + (i_1+1)\sigma_1 P_0(i_1+1, i_2, t)\\
&+ \lambda_1 P_2(i_1, i_2 - 1, t) + (i_1+1)\sigma_1 P_2(i_1+1, i_2-1, t),\\
\frac{\partial P_2(i_1,i_2,t)}{\partial t} &= -(\lambda_1 + \lambda_2 + \mu_2 + i_1\sigma_1)P_2(i_1,i_2,t) + \lambda_2 P_2(i_1, i_2 - 1, t)\\
&+ \lambda_2 P_0(i_1,i_2,t) + (i_2+1)\sigma_2 P_0(i_1, i_2+1, t).
\end{aligned} \tag{2}$$

To obtain the characteristics of a given queuing system, it is necessary to determine the probability distribution (1). In the limit case, i.e., by solving the system of equations (2), it is difficult to accomplish this because the system is non-trivial. Therefore, it is proposed to construct a diffusion process with the help of which one can approximate the probability distribution (1), thus solving the objective.

Definition 1. Let consider as a partial derivative-characteristic function a function of the order

$$H_k(z,u,t) = \sum_{i_1=0}^{\infty} z^{i_1} \sum_{i_2=0}^{\infty} e^{jui_2} P_k(i_1,i_2,t). \tag{3}$$

For further investigations, we pass from the system of equations (2) the system of equations for functions (3). We obtain:

$$\begin{aligned}
\frac{\partial H_0(z,u,t)}{\partial t} &= -(\lambda_1+\lambda_2)H_0(z,u,t) - \sigma_1 z\frac{\partial H_0(z,u,t)}{\partial z} + j\sigma_2\frac{\partial H_0(z,u,t)}{\partial u}\\
&+\mu_1 H_1(z,u,t) + \mu_2 H_2(z,u,t),\\
\frac{\partial H_1(z,u,t)}{\partial t} &= -(\lambda_1+\lambda_2+\mu_1)H_1(z,u,t) + \lambda_1 z H_1(z,u,t) + \lambda_2 e^{ju}H_1(z,u,t)\\
&+\lambda_1 H_0(z,u,t) + \lambda_1 e^{ju}H_2(z,u,t) + \sigma_1\frac{\partial H_0(z,u,t)}{\partial z} + \sigma_1 e^{ju}\frac{\partial H_2(z,u,t)}{\partial z},\\
\frac{\partial H_2(z,u,t)}{\partial t} &= -(\lambda_1+\lambda_2+\mu_2)H_2(z,u,t) + \lambda_2 e^{ju}H_2(z,u,t) + \lambda_2 H_0(z,u,t)\\
&-\sigma_1 z\frac{\partial H_2(z,u,t)}{\partial z} - j\sigma_2 e^{-ju}\frac{\partial H_0(z,u,t)}{\partial u}.
\end{aligned} \tag{4}$$

Summing the equations of system (4), setting $z = 1$ and denoting $H_0(1,u,t) + H_1(1,u,t) + H_2(1,u,t) = H(1,u,t)$, we obtain another additive equation:

$$\frac{\partial H(1,u,t)}{\partial t} = (e^{ju} - 1) \times \left(\lambda_2 H_1(1,u,t) + (\lambda_1+\lambda_2)H_2(1,u,t) + j\sigma_2 e^{-ju}\frac{\partial H_0(1,u,t)}{\partial u} + \sigma_1 \frac{\partial H_2(1,u,t)}{\partial z}\right), \tag{5}$$

which we shall solve jointly with system (4) by the asymptotic-diffusion analysis method.

The diffusion analysis will be performed for the non-priority component in the limit condition of large delay of customers in the second orbit, i.e. at $\sigma_2 \to 0$, in several steps:

(1) performing the first-order asymptotics, we obtain the transfer coefficient of some diffusion process, by applying it to approximate the probability distribution of the customers number on the orbit. Also at this stage, we find expressions for the stationary probability distribution of the server states and for the partial derivative function of the customers number on the first orbit;

(2) performing the second order asymptotics, we obtain the diffusion coefficient of some diffusion process;

(3) at the third stage we obtain an approximation of the customers number probability distribution in the non-priority orbit.

3. ASYMPTOTIC-DIFFUSION ANALYSIS BY NON-PRIORITY COMPONENT

3.1. First Order Asymptotics

In system (4) and equation (5), denoting $\sigma_2 = \epsilon$, we make the following substitutions:

$$\sigma_2 t = \tau, \quad u = \sigma_2\epsilon = \epsilon w, \quad H_k(z,u,t) = F_k(z,w,\tau,\epsilon).$$

Then (4) and (5), taking into account the substitutions, will be described in the form:

$$\begin{aligned}
\epsilon\frac{\partial F_0(z,w,\tau,\epsilon)}{\partial \tau} &= -(\lambda_1+\lambda_2)F_0(z,w,\tau,\epsilon) - \sigma_1 z\frac{\partial F_0(z,w,\tau,\epsilon)}{\partial z} + j\frac{\partial F_0(z,w,\tau,\epsilon)}{\partial w} \\
&\quad + \mu_1 F_1(z,w,\tau,\epsilon) + \mu_2 F_2(z,w,\tau,\epsilon), \\
\epsilon\frac{\partial F_1(z,w,\tau,\epsilon)}{\partial \tau} &= -(\lambda_1+\lambda_2+\mu_1)F_1(z,w,\tau,\epsilon) + \lambda_1 z F_1(z,w,\tau,\epsilon) + \lambda_2 e^{j\epsilon w}F_1(z,w,\tau,\epsilon) \\
&\quad + \lambda_1 F_0(z,w,\tau,\epsilon) + \lambda_1 e^{j\epsilon w}F_2(z,w,\tau,\epsilon) + \sigma_1\frac{\partial F_0(z,w,\tau,\epsilon)}{\partial z} + \sigma_1 e^{j\epsilon w}\frac{\partial F_2(z,w,\tau,\epsilon)}{\partial z}, \\
\epsilon\frac{\partial F_2(z,w,\tau,\epsilon)}{\partial \tau} &= -(\lambda_1+\lambda_2+\mu_2)F_2(z,w,\tau,\epsilon) + \lambda_2 e^{j\epsilon w}F_2(z,w,\tau,\epsilon) + \lambda_2 F_0(z,w,\tau,\epsilon) \\
&\quad - \sigma_1 z\frac{\partial F_2(z,w,\tau,\epsilon)}{\partial z} - je^{-j\epsilon w}\frac{\partial F_0(z,w,\tau,\epsilon)}{\partial w}.
\end{aligned} \tag{6}$$

$$\epsilon\frac{\partial F(1,w,\tau,\epsilon)}{\partial \tau} = (e^{j\epsilon w} - 1) \times \left(\lambda_2 F_1(1,w,\tau,\epsilon) + (\lambda_1+\lambda_2)F_2(1,w,\tau,\epsilon) + je^{-j\epsilon w}\frac{\partial F_0(1,w,\tau,\epsilon)}{\partial w} + \sigma_1\frac{\partial F_2(1,w,\tau,\epsilon)}{\partial z}\right), \tag{7}$$

where $F(1,w,\tau,\epsilon) = F_0(1,w,\tau,\epsilon) + F_1(1,w,\tau,\epsilon) + F_2(1,w,\tau,\epsilon)$.

Formulate the following theorem.

Theorem 1. *Denote the equation system solution* (6):

$$\lim_{\epsilon\to 0} F_k(z,w,\tau,\epsilon) = F_k(z,w,\tau), \quad k=\overline{0,2}.$$

Then the following statement is true

$$F_k(z,w,\tau) = G_k(z,x(\tau))e^{jwx(\tau)}, \quad k=\overline{0,2}. \tag{8}$$

For convenience of notation, we will omit the argume τ*:* $x(\tau)=x$*. Functions* $G_k(z,x)$*,* $k=\overline{0,2}$ *– are partial derivative functions of the customers number on the first orbit, which have the form*

$$G_0(z,x) = \frac{(\mu_1-\lambda_1)^{\frac{\lambda_1}{\sigma_1}+1}}{\mu_1(\mu_1-\lambda_1 z)^{\frac{\lambda_1}{\sigma_1}}} - \frac{\lambda_2+x}{\sigma_1} z^{-\frac{\lambda_1+\lambda_2+\mu_2+x}{\sigma_1}} \int\limits_0^z y^{\frac{\lambda_1+\lambda_2+\mu_2+x-\sigma_1}{\sigma_1}} \frac{(\mu_1-\lambda_1)^{\frac{\lambda_1}{\sigma_1}+1}}{\mu_1(\mu_1-\lambda_1 y)^{\frac{\lambda_1}{\sigma_1}}} dy,$$

$$G_1(z) = \left(\frac{\mu_1-\lambda_1}{\mu_1-\lambda_1 z}\right)^{\frac{\lambda_1+\sigma_1}{\sigma_1}} \frac{\lambda_1}{\mu_1}, \tag{9}$$

$$G_2(z,x) = \frac{\lambda_2+x}{\sigma_1} z^{-\frac{\lambda_1+\lambda_2+\mu_2+x}{\sigma_1}} \int\limits_0^z y^{\frac{\lambda_1+\lambda_2+\mu_2+x-\sigma_1}{\sigma_1}} \frac{(\mu_1-\lambda_1)^{\frac{\lambda_1}{\sigma_1}+1}}{\mu_1(\mu_1-\lambda_1 y)^{\frac{\lambda_1}{\sigma_1}}} dy,$$

value $x(\tau)$ *is the differential equation solution*

$$x'(\tau) = -x(\tau)G_0(1,x) + \lambda_2 G_1(1) + (\lambda_1+\lambda_2)G_2(1,x) + \sigma_1 \left.\frac{\partial G_2(z,x)}{\partial z}\right|_{z=1}. \tag{10}$$

Proof. In the system (6) we perform the limit transition at $\epsilon \to 0$, obtain

$$\begin{gathered}
-(\lambda_1+\lambda_2)F_0(z,w,\tau) - \sigma_1 z\frac{\partial F_0(z,w,\tau)}{\partial z} + j\frac{\partial F_0(z,w,\tau)}{\partial w} \\
+\mu_1 F_1(z,w,\tau) + \mu_2 F_2(z,w,\tau) = 0, \\
-(\lambda_1+\mu_1)F_1(z,w,\tau) + \lambda_1 z F_1(z,w,\tau) \\
+\lambda_1 F_0(z,w,\tau) + \lambda_1 F_2(z,w,\tau) + \sigma_1\frac{\partial F_0(z,w,\tau)}{\partial z} + \sigma_1\frac{\partial F_2(z,w,\tau)}{\partial z} = 0, \\
-(\lambda_1+\mu_2)F_2(z,w,\tau) + \lambda_2 F_0(z,w,\tau) - \sigma_1 z\frac{\partial F_2(z,w,\tau)}{\partial z} - j\frac{\partial F_0(z,w,\tau)}{\partial w} = 0.
\end{gathered} \tag{11}$$

The solution of the equations system (11) will be found in the form (8). Then (11) will be rewritten as

$$\begin{gathered}
-(\lambda_1+\lambda_2+x(\tau))G_0(z,x) - \sigma_1 z\frac{\partial G_0(z,x)}{\partial z} + \mu_1 G_1(z) + \mu_2 G_2(z,x) = 0, \\
-(\lambda_1(1-z)+\mu_1)G_1(z) + \lambda_1 G_0(z,x) + \lambda_1 G_2(z,x) + \sigma_1\frac{\partial G_0(z,x)}{\partial z} + \sigma_1\frac{\partial G_2(z,x)}{\partial z} = 0, \\
-(\lambda_1+\mu_2)G_2(z,x) + (\lambda_2+x(\tau))G_0(z,x) - \sigma_1 z\frac{\partial G_2(z,x)}{\partial z} = 0.
\end{gathered} \tag{12}$$

Let's introduce the notation $G_{02}(z,x) = G_0(z,x) + G_2(z,x)$. Then by differentiating $G_{02}(z,x)$ by z, we obtain $\frac{\partial G_{02}(z,x)}{\partial z} = \frac{\partial G_0(z,x)}{\partial z} + \frac{\partial G_2(z,x)}{\partial z}$.

Sum the first and the third equation (12) and make a system of two equations (the sum of the first and the third equations and the second equation (12)). Taking into account the introduced notations, we obtain

$$\begin{aligned} -\lambda_1 G_{02}(z,x) + \mu_1 G_1(z) - \sigma_1 z \frac{\partial G_{02}(z,x)}{\partial z} = 0, \\ -(\lambda_1(1-z)+\mu_1)G_1(z) + \lambda_1 G_{02}(z,x) + \sigma_1 \frac{\partial G_{02}(z,x)}{\partial z} = 0. \end{aligned} \tag{13}$$

Multiplying the second equation by z and summing the equations of the system, we obtain

$$\lambda_1 G_{02}(z,x) - \mu_1 G_1(z) + \lambda_1 z G_1(z) = 0. \tag{14}$$

In (14), express $G_1(z)$ through $G_{02}(z,x)$ and substitute it into the first equation of system (13) to obtain a homogeneous differential equation with reference to the function $G_{02}(z,x) = G_{02}(z)$:

$$\frac{\lambda_1^2 z}{\mu_1 - \lambda_1 z} G_{02}(z) - \sigma_1 z \frac{\partial G_{02}(z)}{\partial z} = 0,$$

whose solution is

$$G_{02}(z) = (\mu_1 - \lambda_1 z)^{-\frac{\lambda_1}{\sigma_1}} C. \tag{15}$$

For the function $G_1(z)$ we similarly obtain the differential equation

$$(\lambda_1 + \sigma_1)G_1(z) + \left(\sigma_1 z - \frac{\sigma_1 \mu_1}{\lambda_1}\right) \frac{\partial G_1(z)}{\partial z} = 0,$$

whose solution is

$$G_1(z) = (\mu_1 - \lambda_1 z)^{-\frac{\lambda_1+\sigma_1}{\sigma_1}} C. \tag{16}$$

We shall seek the constant C. In functions $G_k(z,x)$, $k = \overline{0,2}$, we will assume $z = 1$ and denote $G_0(1,x) = R_0(x)$, $G_1(1) = R_1$, $G_2(1,x) = R_2(x)$. The values of $R_0(x), R_1, R_2(x)$ satisfy the normalization condition $R_0(x) + R_1 + R_2(x) = 1$. If we switch to stationary mode in equation (10) and denote $x = \kappa$ by the solution of the stationary equation, we obtain the stationary probability distribution of the server states $R_0(\kappa), R_1, R_2(\kappa)$.

In equation (14) we assume $z = 1$ and, adding the normalization condition, we obtain the system of equations

$$\begin{aligned} \lambda_1(R_0(x) + R_2(x)) - \mu_1 R_1 + \lambda_1 R_1 = 0, \\ R_0(x) + R_1 + R_2(x) = 1. \end{aligned}$$

From where we obtain

$$R_1 = \frac{\lambda_1}{\mu_1}, \qquad R_0(x) + R_2(x) = \frac{\mu_1 - \lambda_1}{\mu_1}.$$

Then assuming $z = 1$ in (15) and (16) and using the found $R_k(x)$, we obtain

$$G_1(z) = \left(\frac{\mu_1 - \lambda_1}{\mu_1 - \lambda_1 z}\right)^{\frac{\lambda_1+\sigma_1}{\sigma_1}} \frac{\lambda_1}{\mu_1}, \qquad G_{02}(z) = \left(\frac{\mu_1 - \lambda_1}{\mu_1 - \lambda_1 z}\right)^{\frac{\lambda_1}{\sigma_1}} \frac{\mu_1 - \lambda_1}{\mu_1}.$$

Consider the third equation of the system (12) and found $G_{02}(z) = G_{02}(z,x) = G_0(z,x) + G_2(z,x)$:

$$\begin{aligned} -(\lambda_1 + \mu_2)G_2(z,x) + (\lambda_2 + x(\tau))G_0(z,x) - \sigma_1 z \frac{\partial G_2(z,x)}{\partial z} = 0, \\ G_0(z,x) + G_2(z,x) = \left(\frac{\mu_1 - \lambda_1}{\mu_1 - \lambda_1 z}\right)^{\frac{\lambda_1}{\sigma_1}} \frac{\mu_1 - \lambda_1}{\mu_1}. \end{aligned} \tag{17}$$

Expressing $G_0(z,x)$ from the second equation and substituting it into the first system equation (17), we obtain an inhomogeneous differential equation with regerence to the function $G_2(z,x)$:

$$(G_2(z,x))'_z + \frac{\lambda_1+\lambda_2+\mu_2+x}{\sigma_1 z}G_2(z,x) = \frac{\lambda_2+x}{\sigma_1 z}\left(\frac{\mu_1-\lambda_1}{\mu_1-\lambda_1 z}\right)^{\frac{\lambda_1}{\sigma_1}}\frac{\mu_1-\lambda_1}{\mu_1},$$

whose solution is

$$G_2(z,x) = \frac{\lambda_2+x}{\sigma_1}z^{-\frac{\lambda_1+\lambda_2+\mu_2+x}{\sigma_1}}\int\limits_0^z y^{\frac{\lambda_1+\lambda_2+\mu_2+x-\sigma_1}{\sigma_1}}\frac{(\mu_1-\lambda_1)^{\frac{\lambda_1}{\sigma_1}+1}}{\mu_1(\mu_1-\lambda_1 y)^{\frac{\lambda_1}{\sigma_1}}}dy.$$

Substituting the found expression for the function $G_2(z,x)$ into the system (17), we obtain the solution for the function $G_0(z,x)$:

$$G_0(z,x) = \frac{(\mu_1-\lambda_1)^{\frac{\lambda_1}{\sigma_1}+1}}{\mu_1(\mu_1-\lambda_1 z)^{\frac{\lambda_1}{\sigma_1}}} - \frac{\lambda_2+x}{\sigma_1}z^{-\frac{\lambda_1+\lambda_2+\mu_2+x}{\sigma_1}}\int\limits_0^z y^{\frac{\lambda_1+\lambda_2+\mu_2+x-\sigma_1}{\sigma_1}}\frac{(\mu_1-\lambda_1)^{\frac{\lambda_1}{\sigma_1}+1}}{\mu_1(\mu_1-\lambda_1 y)^{\frac{\lambda_1}{\sigma_1}}}dy.$$

The expressions found for the functions $G_k(z,x),\ k=0,1,2$ coincide with (9).

Consider equation (7). Decompose the exponent into a Taylor series

$$e^{j\epsilon w} = 1 + j\epsilon w + O(\epsilon^2)$$

and divide the left and right parts of (7) by $j\epsilon w$. Then we execute the limit transition at $\epsilon \to 0$, we obtain

$$\frac{\partial F(1,w,\tau)}{jw\partial\tau} = \lambda_2 F_1(1,w,\tau) + (\lambda_1+\lambda_2)F_2(1,w,\tau) + j\frac{\partial F_0(1,w,\tau)}{\partial w} + \sigma_1\frac{\partial F_2(1,w,\tau)}{\partial z}.$$

Execute substitutions

$$F_k(1,w,\tau) = G_k(1,x)e^{jwx} = R_k(x)e^{jwx},\ k=0,1,2, \qquad F(1,w,\tau) = G(1,x)e^{jwx} = e^{jwx}.$$

Then we obtain the following differential equation for $x(\tau)$:

$$x'(\tau) = -x(\tau)G_0(1,x) + \lambda_2 G_1(1) + (\lambda_1+\lambda_2)G_2(1,x) + \sigma_1\left.\frac{\partial G_2(z,x)}{\partial z}\right|_{z=1},$$

which coincides with (10). Thus, Theorem 1 is proved.

Denoting:

$$\left.\frac{\partial G_k(z,x)}{\partial z}\right|_{z=1} = \frac{\partial G_k(1,x)}{\partial z},\ k=0,1,2, \qquad \left.\frac{\partial G_{02}(z)}{\partial z}\right|_{z=1} = \frac{\partial G_{02}(1)}{\partial z}.$$

Denote the right-hand side of equation (10) by $a(x)$ and simplify it using (9); we obtain

$$a(x) = x'(\tau) = -x(\tau)G_0(1,x) + \lambda_2 G_1(1) + (\lambda_1+\lambda_2)G_2(1,x) + \sigma_1\frac{\partial G_2(1,x)}{\partial z} = \lambda_2 - \mu_2 R_2(x).$$

The value $a(x)$ has the meaning of the transfer coefficient of some diffusion process, by which we obtain an approximation of the calls number probability distribution on the orbit.

Corollary 1. *The ergodicity condition of the considered RQ-system has the following form*

$$\frac{\lambda_1}{\mu_1} + \frac{\lambda_2}{\mu_2} < 1.$$

Proof. A sufficient condition of the system ergodicity is the inequality

$$\lim_{x\to\infty} a(x) < 0.$$

Thus, we obtain

$$\lim_{x\to\infty} a(x) = \lim_{x\to\infty} (\lambda_2 - \mu_2 R_2(x)) = \lambda_2 - \mu_2\Big(\frac{\mu_1-\lambda_1}{\mu_1}\Big) < 0.$$

From where we obtain the condition

$$\frac{\lambda_1}{\mu_1} + \frac{\lambda_2}{\mu_2} < 1.$$

Corollary 2. *The derivative function of the number of calls in a priority orbit has the following form*

$$G(z) = \frac{\lambda_1}{\mu_1}\left(\frac{\mu_1-\lambda_1}{\mu_1-\lambda_1 z}\right)^{\frac{\lambda_1+\sigma_1}{\sigma_1}} + \frac{\mu_1-\lambda_1}{\mu_1}\left(\frac{\mu_1-\lambda_1}{\mu_1-\lambda_1 z}\right)^{\frac{\lambda_1}{\sigma_1}}.$$

Thus, the number of calls probability distribution on the first orbit has the form of a weighted sum of two negative-binomial distributions with weights $\frac{\lambda_1}{\mu_1}$ and $\frac{\mu_1-\lambda_1}{\mu_1}$.

3.2. Second Order Asymptotics

Consider the characteristic function of a random process $i(t) - \frac{x(\sigma_2 t)}{\sigma_2}$. For this purpose, in system (4) and equation (5) we make a substitution:

$$H_k(z,u,t) = e^{\frac{jux(\sigma_2 t)}{\sigma_2}} H_k^{(2)}(z,u,t), \quad k = 0,1,2.$$

Denoting $\sigma_2 = \epsilon^2$, in the equations system for $H_k^{(2)}(z,u,t), \quad k=0,1,2$ we substitute

$$\tau = \epsilon^2 t, \quad u = \epsilon w, \quad H_k^{(2)}(z,u,t) = F_k^{(2)}(z,w,\tau,\epsilon),$$

we obtain a system for functions $F_k^{(2)}(z,w,\tau,\epsilon), k=0,1,2$ and an additive equation

$$\begin{aligned}
&\epsilon^2\frac{\partial F_0^{(2)}(z,w,\tau,\epsilon)}{\partial\tau} + j\epsilon w a(x) F_0^{(2)}(z,w,\tau,\epsilon) = -(\lambda_1+\lambda_2+x)F_0^{(2)}(z,w,\tau,\epsilon)\\
&-\sigma_1 z\frac{\partial F_0^{(2)}(z,w,\tau,\epsilon)}{\partial z} + j\epsilon\frac{\partial F_0^{(2)}(z,w,\tau,\epsilon)}{\partial w} + \mu_1 F_1^{(2)}(z,w,\tau,\epsilon) + \mu_2 F_2^{(2)}(z,w,\tau,\epsilon),\\
&\epsilon^2\frac{\partial F_1^{(2)}(z,w,\tau,\epsilon)}{\partial\tau} + j\epsilon w a(x) F_1^{(2)}(z,w,\tau,\epsilon) = -(\lambda_1+\lambda_2+\mu_1)F_1^{(2)}(z,w,\tau,\epsilon)\\
&+\lambda_1 z F_1^{(2)}(z,w,\tau,\epsilon) + \lambda_2 e^{j\epsilon w}F_1^{(2)}(z,w,\tau,\epsilon) + \lambda_1 F_0^{(2)}(z,w,\tau,\epsilon)\\
&+\lambda_1 e^{j\epsilon w}F_2^{(2)}(z,w,\tau,\epsilon) + \sigma_1\frac{\partial F_0^{(2)}(z,w,\tau,\epsilon)}{\partial z} + \sigma_1 e^{j\epsilon w}\frac{\partial F_2^{(2)}(z,w,\tau,\epsilon)}{\partial z},\\
&\epsilon^2\frac{\partial F_2^{(2)}(z,w,\tau,\epsilon)}{\partial\tau} + j\epsilon w a(x) F_2^{(2)}(z,w,\tau,\epsilon) = -(\lambda_1+\lambda_2+\mu_2)F_2^{(2)}(z,w,\tau,\epsilon)\\
&+\lambda_2 e^{j\epsilon w}F_2^{(2)}(z,w,\tau,\epsilon) + \lambda_2 F_0^{(2)}(z,w,\tau,\epsilon)\\
&-\sigma_1 z\frac{\partial F_2^{(2)}(z,w,\tau,\epsilon)}{\partial z} - je^{-j\epsilon w}\frac{\partial F_0^{(2)}(z,w,\tau,\epsilon)}{\partial w} + xe^{-j\epsilon w}F_0^{(2)}(z,w,\tau,\epsilon).
\end{aligned} \tag{18}$$

$$\begin{aligned}
&\epsilon^2\frac{\partial F^{(2)}(1,w,\tau,\epsilon)}{\partial\tau} + j\epsilon w a(x)F^{(2)}(1,w,\tau,\epsilon) = (e^{j\epsilon w}-1)\\
&\times\Big(-xe^{-j\epsilon w}F_0^{(2)}(1,w,\tau,\epsilon) + \lambda_2 F_1^{(2)}(1,w,\tau,\epsilon) + (\lambda_1+\lambda_2)F_2^{(2)}(1,w,\tau,\epsilon)\\
&+j\epsilon e^{-j\epsilon w}\frac{\partial F_0^{(2)}(1,w,\tau,\epsilon)}{\partial w} + \sigma_1\frac{\partial F_2^{(2)}(1,w,\tau,\epsilon)}{\partial z}\Big).
\end{aligned} \tag{19}$$

Denote: $F_k^{(2)}(z,w,\tau) = \lim\limits_{\epsilon\to 0} F_k^{(2)}(z,w,\tau,\epsilon)$, $k = 0,1,2$. We formulate and prove the following theorem.

Theorem 2. *The function* $F_k^{(2)}(z,w,\tau), k = 0,1,2$ *have the form* $F_k^{(2)}(z,w,\tau) = \Phi(w,\tau)G_k(z,x)$, *where the function* $\Phi(w,\tau)$ *is a characteristic function of process* $y(\tau) = \lim\limits_{\sigma_2\to 0}\sqrt{\sigma_2}(i(\frac{\tau}{\sigma_2}) - \frac{x(\tau)}{\sigma_2})$,

and satisfies the differential equation

$$\frac{\partial\Phi(w,\tau)}{\partial\tau} = a'(x)w\frac{\partial\Phi(w,\tau)}{\partial w} - \frac{w^2}{2}b(x)\Phi(w,\tau), \tag{20}$$

where

$$a(x) = \lambda_2 - \mu_2 R_2(x), \tag{21}$$

$$b(x) = a(x) + 2((\lambda_2 - a(x))R_2(x) - \mu_2 h_2(x)). \tag{22}$$

Here $h_2(x) = h_2(1,x)$,

$$h_2(1,x) = \frac{1}{\sigma_1}\int\limits_0^1 y^{\frac{\lambda_1+\lambda_2+\mu_2+x-\sigma_1}{\sigma_1}} \cdot D(y,x)dy, \tag{23}$$

$$D(z,x) = (\lambda_2 + x)h_{02}(z,x) + (\lambda_2 - a(x))G_2(z,x) - xG_0(z,x), \tag{24}$$

$$h_{02}(z,x) = \left(\frac{\mu_1-\lambda_1}{\mu_1 - z\lambda_1}\right)^{\frac{\lambda_1}{\sigma_1}} \cdot \left(\frac{1}{\mu_1}A(1,x) - \frac{1}{\sigma_1}\int\limits_z^1 \left(\frac{\mu_1-\lambda_1}{\mu_1-y\lambda_1}\right)^{\frac{-\lambda_1}{\sigma_1}} B(y,x)dy\right), \tag{25}$$

$$\begin{aligned} A(1,x) = a(x)\frac{\partial G_{02}(1)}{\partial z} - (\lambda_1+\lambda_2+\sigma_1)\frac{\partial G_2(1,x)}{\partial z} + x\frac{\partial G_0(1,x)}{\partial z} - \lambda_1 R_2(x) \\ +(a(x)-\lambda_2)R_1 + (a(x)-\lambda_2)\frac{\partial G_1(1)}{\partial z} - \sigma_1\frac{\partial^2 G_2(1,x)}{\partial z^2}, \end{aligned} \tag{26}$$

$$\begin{aligned} B(z,x) = (a(x)-\lambda_2)G_1(z) - \lambda_1 G_2(z,x) - \sigma_1\frac{\partial G_2(z,x)}{\partial z} - \frac{(z-1)\lambda_1 - \mu_1}{(z\lambda_1-\mu_1)(z-1)} \\ \times\left(a(x)G_{02}(z) - (\lambda_2+z\lambda_1)G_2(z,x) + xG_0(z,x) + z(a(x)-\lambda_2)G_1(z) - z\sigma_1\frac{\partial G_2(z,x)}{\partial z}\right). \end{aligned} \tag{27}$$

Proof. The solution of the equation system (18) is written in the following form:

$$F_k(z,w,\tau,\epsilon) = \Phi(w,\tau)\left(G_k(z,x) + j\epsilon w g_k(z,x)\right) + O(\epsilon^2), \quad k = 0,1,2.$$

Substituting these functions into the system of equation (18), making simple transformations and performing the limit transition at $\epsilon \to 0$, we obtain a system of equations:

$$\begin{aligned} a(x)G_0(z,x) = -(\lambda_1+\lambda_2+x)g_0(z,x) + \mu_1 g_1(z,x) + \mu_2 g_2(z,x) \\ -\sigma_1 z\frac{\partial g_0(z,x)}{\partial z} + G_0(z,x)\frac{\partial\Phi(w,\tau)}{w\Phi(w,\tau)\partial w}, \\ a(x)G_1(z) = -(\lambda_1(1-z)+\mu_1)g_1(z,x) + \lambda_1 g_0(z,x) + \lambda_1 g_2(z,x) + \lambda_2 G_1(z,x) \\ +\lambda_1 G_2(z,x) + \sigma_1\frac{\partial g_0(z,x)}{\partial z} + \sigma_1\frac{\partial g_2(z,x)}{\partial z} + \sigma_1\frac{\partial G_2(z,x)}{\partial z}, \\ a(x)G_2(z,x) = -(\lambda_1+\mu_2)g_2(z,x) + (\lambda_2+x)g_0(z,x) + \lambda_2 G_2(z,x) \\ -xG_0(z,x) - \sigma_1 z\frac{\partial g_2(z,x)}{\partial z} - G_0(z,x)\frac{\partial\Phi(w,\tau)}{w\Phi(w,\tau)\partial w}. \end{aligned} \tag{28}$$

The functions $g_k(z,x), k=0,1,2$ will be solved as the sum of a general homogeneous solution and a partial solution of an inhomogeneous differential equation:

$$g_k(z,x) = CG_k(z,x) + h_k(z,x) - \phi_k(z,x)\frac{\partial\Phi(w,\tau)}{w\Phi(w,\tau)\partial w},\ k=0,1,2.$$

Substituting $g_k(z,x)$ into (28),we can confirm that the coefficient in front of the constant C is zero. Then equating the coefficients at $\frac{\partial\Phi(w,\tau)}{w\Phi(w,\tau)\partial w}$, we obtain the equations system to find functions $\phi_k(z,x)$. If the equation system for the functions $G_k(z,x)$ is differentiated by the variable x, we can confirm that it coincides with the equation system for finding the functions $\phi_k(z,x)$. Thus, we conclude that

$$\phi_k(z,x) = \frac{\partial G_k(z,x)}{\partial x},\ k=0,1,2.$$

Equating the remaining summands, we can write the equation system for determining the functions $h_k(z,x), k=0,1,2$:

$$\begin{aligned}
-(\lambda_1+\lambda_2+x)h_0(z,x) + \mu_1 h_1(z,x) + \mu_2 h_2(z,x) - \sigma_1 z\frac{\partial h_0(z,x)}{\partial z} &= a(x)G_0(z,x),\\
\lambda_1 h_0(z,x) - (\lambda_1(1-z)+\mu_1)h_1(z,x) + \lambda_1 h_2(z,x) + \sigma_1\frac{\partial h_0(z,x)}{\partial z} + \sigma_1\frac{\partial h_2(z,x)}{\partial z}&\\
= (a(x)-\lambda_2)G_1(z) - \sigma_1\frac{\partial G_2(z,x)}{\partial z} - \lambda_1 G_2(z,x),&\\
(\lambda_2+x)h_0(z,x) - (\lambda_1+\mu_2)h_2(z,x) - \sigma_1 z\frac{\partial h_2(z,x)}{\partial z} &= (a(x)-\lambda_2)G_2(z,x) + xG_0(z,x).
\end{aligned} \tag{29}$$

Sum the first and the third system equation (29) and add the second equation to the resulting equation to obtain the system:

$$\begin{aligned}
-\lambda_1(h_0(z,x)+h_2(z,x)) + \mu_1 h_1(z,x) - \sigma_1 z\left(\frac{\partial h_0(z,x)}{\partial z} + \frac{\partial h_2(z,x)}{\partial z}\right)&\\
= a(x)(G_0(z,x)+G_2(z,x)) - \lambda_2 G_2(z,x) + xG_0(z,x),&\\
\lambda_1(h_0(z,x)+h_2(z,x)) - (\lambda_1(1-z)+\mu_1)h_1(z,x) + \sigma_1\left(\frac{\partial h_0(z,x)}{\partial z} + \frac{\partial h_2(z,x)}{\partial z}\right) =&\\
= (a(x)-\lambda_2)G_1(z) - \sigma_1\frac{\partial G_2(z,x)}{\partial z} - \lambda_1 G_2(z,x).&
\end{aligned}$$

Denote:

$$h_{02}(z,x) = h_0(z,x) + h_2(z,x).$$

Then the equation system will be rewritten in the form:

$$\begin{aligned}
-\lambda_1 h_{02}(z,x) + \mu_1 h_1(z,x) - \sigma_1 z\frac{\partial h_{02}(z,x)}{\partial z}&\\
= a(x)(G_0(z,x)+G_2(z,x)) - \lambda_2 G_2(z,x) + xG_0(z,x),&\\
\lambda_1 h_{02}(z,x) - (\lambda_1(1-z)+\mu_1)h_1(z,x) + \sigma_1\frac{\partial h_{02}(z,x)}{\partial z}&\\
= (a(x)-\lambda_2)G_1(z) - \sigma_1\frac{\partial G_2(z,x)}{\partial z} - \lambda_1 G_2(z,x).&
\end{aligned} \tag{30}$$

Multiply the second equation by z and sum the system equations, we obtain:

$$\begin{aligned}\lambda_1 h_{02}(z,x) - (\mu_1 - \lambda_1 z)h_1(z,x) = \frac{1}{z-1}\Big(a(x)(G_0(z,x) + G_2(z,x) + zG_1(z))\\ +xG_0(z,x) - \lambda_2 z G_1(z) - (\lambda_1 z + \lambda_2)G_2(z,x) - \sigma_1 z\frac{\partial G_2(z,x)}{\partial z}\Big).\end{aligned} \tag{31}$$

Denote the right side of the equation by $A(z,x)$ and seek the limit at $z \to 1$:

$$\begin{aligned}&\lim_{z\to 1}\frac{(a(x)+x)G_0(z,x) + z(a(x)-\lambda_2)G_1(z) - (\lambda_1 z + \lambda_2 - a(x))G_2(z,x) - \sigma_1 z\frac{\partial G_2(z,x)}{\partial z}}{z-1}\\ &= \lim_{z\to 1}\left(a(x)\frac{\partial G_{02}(z)}{\partial z} - (\lambda_1+\lambda_2+\sigma_1)\frac{\partial G_2(z,x)}{\partial z} + x\frac{\partial G_0(z,x)}{\partial z} - \lambda_1 G_2(z,x)\right.\\ &\left. +(a(x)-\lambda_2)G_1(z) + (a(x)-\lambda_2)\frac{\partial G_1(z)}{\partial z} - \sigma_1 z\frac{\partial^2 G_2(z,x)}{\partial z^2}\right)\\ &= \lim_{z\to 1} A(z,x) = a(x)\frac{\partial G_{02}(1)}{\partial z} - (\lambda_1+\lambda_2+\sigma_1)\frac{\partial G_2(1,x)}{\partial z} + x\frac{\partial G_0(1,x)}{\partial z} - \lambda_1 R_2(x)\\ &+(a(x)-\lambda_2)R_1 + (a(x)-\lambda_2)\frac{\partial G_1(1)}{\partial z} - \sigma_1\frac{\partial^2 G_2(1,x)}{\partial z^2} = A(1,x),\end{aligned}$$

which coincides with (26).

Then equation (31) can be written in the form

$$\lambda_1 h_{02}(z,x) - (\mu_1 - \lambda_1 z)h_1(z,x) = A(1,x).$$

Then we express the function $h_1(z,x)$ and substitute into the second equation of the system (30), we obtain an inhomogeneous differential equation with reference to the function $h_{02}(z,x)$. Denote the right part of the equation by $B(z,x)$:

$$\begin{aligned}B(z,x) = (a(x)-\lambda_2)G_1(z) - \lambda_1 G_2(z,x) - \sigma_1\frac{\partial G_2(z,x)}{\partial z} - \frac{(z-1)\lambda_1 - \mu_1}{(z\lambda_1-\mu_1)(z-1)}\\ \times\left(a(x)G_{02}(z) - (\lambda_2 + z\lambda_1)G_2(z,x) + xG_0(z,x) + z(a(x)-\lambda_2)G_1(z) - z\sigma_1\frac{\partial G_2(z,x)}{\partial z}\right).\end{aligned}$$

Then the differential equation is written in the following form:

$$\lambda_1\left(1 - \frac{\lambda_1(1-z)+\mu_1}{\mu_1 - \lambda_1 z}\right)h_{02}(z,x) + \sigma_1\frac{\partial h_{02}(z,x)}{\partial z} = B(z,x).$$

Its solution has the form

$$h_{02}(z,x) = \left(\frac{\mu_1-\lambda_1}{\mu_1 - z\lambda_1}\right)^{\frac{\lambda_1}{\sigma_1}}\left(\frac{1}{\mu_1}A(1,x) - \frac{1}{\sigma_1}\int_z^1\left(\frac{\mu_1-\lambda_1}{\mu_1-y\lambda_1}\right)^{\frac{-\lambda_1}{\sigma_1}}B(y,x)dy\right),$$

which coincides with (25).

Since $h_0(z,x) = h_{02}(z,x) - h_2(z,x)$, we substitute this into the third equation of the system (29) and denote the right side as

$$D(z,x) = (\lambda_2 + x)h_{02}(z,x) + (\lambda_2 - a(x))G_2(z,x) - xG_0(z,x),$$

obtain a differential equation with reference to $h_2(z,x)$:

$$(\lambda_1+\lambda_2+\mu_2+x)h_2(z,x) + \sigma_1 z\frac{\partial h_2(z,x)}{\partial z} = D(z,x),$$

the solution of which has the form

$$h_2(z,x) = z^{-\frac{\lambda_1+\lambda_2+\mu_2+x}{\sigma_1}} \frac{1}{\sigma_1} \int\limits_0^z y^{\frac{\lambda_1+\lambda_2+\mu_2+x-\sigma_1}{\sigma_1}} D(y,x)dy.$$

Assuming $z = 1$, we obtain (23).

Consider equation (19). Substitute into it

$$F_k(z,w,\tau,\epsilon) = \Phi(w,\tau)\left(G_k(z,x) + j\epsilon w g_k(z,x)\right) + O(\epsilon^2), \ k = 0,1,2,$$

making simple transformations and performing the limit transition at $\epsilon \to 0$, we obtain

$$-\frac{\partial\Phi(w,\tau)}{w^2\Phi(w,\tau)\partial\tau} + a(x)g(1,x) = \lambda_2 g_1(1,x) + (\lambda_1+\lambda_2)g_2(1,x)$$
$$+G_0(1,x)\frac{\partial\Phi(w,\tau)}{w\Phi(w,\tau)\partial w} - xg_0(1,x) + xG_0(1,x) + \sigma_1\frac{\partial g_2(1,x)}{\partial z} + \frac{1}{2}a(x).$$

The functions $g_k(1,x)$ are written in the form

$$g_k(1,x) = C\cdot G_k(1,x) + h_k(1,x) - \phi_k(1,x)\frac{\partial\Phi(w,\tau)}{w\Phi(w,\tau)\partial w}, \quad k = 0,1,2.$$

Considering the conditions $\sum_{k=0}^2 h_k(z,x)\Big|_{z=1} = 0$, $\sum_{k=0}^2 \phi_k(z,x)\Big|_{z=1} = 0$ we obtain the equation

$$\frac{\partial\Phi(w,\tau)}{\partial\tau} = \left(\lambda_2\phi_1(1,x) + (\lambda_1+\lambda_2)\phi_2(1,x) - G_0(1,x) - x\phi_0(1,x) + \sigma_1\frac{\partial\phi_2(1,x)}{\partial z}\right)$$
$$\times w\frac{\partial\Phi(w,\tau)}{\partial w} - \frac{w^2}{2}\Phi(w,\tau)$$
$$\times\left(a(x) + 2\left(\lambda_2 h_1(1,x) + (\lambda_1+\lambda_2)h_2(1,x) - xh_0(1,x) + xG_0(1,x) + \sigma_1\frac{\partial h_2(1,x)}{\partial z}\right)\right).$$

Convert the coefficients in front of $w\frac{\partial\Phi(w,\tau)}{\partial w}$ and $\frac{w^2}{2}\Phi(w,\tau)$, we obtain

$$\lambda_2\phi_1(1,x) + (\lambda_1+\lambda_2)\phi_2(1,x) - G_0(1,x) - x\phi_0(1,x) + \sigma_1\frac{\partial\phi_2(1,x)}{\partial z} = a'(x),$$
$$a(x) + 2\left(\lambda_2 h_1(1,x) + (\lambda_1+\lambda_2)h_2(1,x) - xh_0(1,x) + xG_0(1,x) + \sigma_1\frac{\partial h_2(1,x)}{\partial z}\right)$$
$$= a(x) + 2((\lambda_2 - a(x))R_2(x) - \mu_2 h_2(1,x)) = b(x).$$

Then we obtain the equation

$$\frac{\partial\Phi(w,\tau)}{\partial\tau} = a'(x)w\frac{\partial\Phi(w,\tau)}{\partial w} - \frac{w^2}{2}b(x)\Phi(w,\tau).$$

The theorem is proved.

The function $b(x)$ has the meaning of the diffusion coefficient of some diffusion process, using which we could approximate the probability distribution of the number of calls on the orbit.

3.3. Diffusion Approximation of Probability Distribution of the Number of Calls on a Non-Priority Orbit

Apply the inverse Fourier transform to equation (20). We obtain the Fokker-Planck equation for the probability distribution density function $P(y,\tau) = \frac{\partial P\{y(\tau)<y\}}{\partial y}$:

$$\frac{\partial P(y,\tau)}{\partial \tau} = -\frac{\partial \{ya'(x)P(y,\tau)\}}{\partial y} + \frac{1}{2}\frac{\partial^2 \{b(x)P(y,\tau)\}}{\partial y^2}.$$

It can be concluded that $y(\tau) = \lim\limits_{\sigma_2 \to 0} \sqrt{\sigma_2}\left(i\left(\frac{\tau}{\sigma_2}\right) - \frac{x(\tau)}{\sigma_2}\right)$ is the solution of the stochastic differential equation

$$dy(\tau) = a'(x)yd\tau + \sqrt{b(x)}dW(\tau),$$

where $W(\tau)$ is a Wiener random process, $a'(x)y$ – transfer coefficient, $\sqrt{b(x)}$ – diffusion coefficient.

Consider the diffusion process $z(\tau) = x(\tau) + \sqrt{\sigma_2}y(\tau)$. Note that $z(\tau) = \lim\limits_{\sigma_2 \to 0} \sigma_2 i\left(\frac{\tau}{\sigma_2}\right)$. Denote by $V(z)$ the stationary probability distribution density function of the process $z(\tau)$. It may be shown that the density $V(z)$ has the form

$$V(z) = \frac{C}{b(z)} \cdot \exp\left(\frac{2}{\sigma_2}\int\limits_0^z \frac{a(x)}{b(x)}dx\right).$$

A more detailed description of the procedure for constructing the diffusion approximation and finding the kind of density $V(z)$ can be found in [13, 14].

To construct a diffusion approximation of the probability distribution of the number of requests on a non-priority orbit, we will use the formula:

$$P(i) = \frac{V(i\sigma_2)}{\sum\limits_{n=0}^{\infty} V(n\sigma_2)}. \tag{32}$$

Thus, there is no need to determine the value of the constant C.

4. THE DIFFUSION APPROXIMATION ALGORITHM IN MATHCAD SOFTWARE

The obtained theoretical results were realized in MathCAD software package. The numerical realization algorithm for finding the diffusion approximation of the number of requests on the non-priority orbit is given below.

Algorithm 1.

(1) set system parameter: $\lambda_1, \lambda_2, \mu_1, \mu_2, \sigma_1, \sigma_2$. Set a sufficiently large number of N;
(2) by formulas (9) write down the functions $G_k(z,x), k = 0,1,2$ and $R_k(x) = G_k(1,x), k = 0,1,2$;
(3) by formula (21) write down the function $a(x)$ and find the solution κ of the stationary equation $a(x) = 0$, using the built-in function of MathCAD software

$$\kappa := root(a(x), x, 0, N);$$

(4) determine the values $R_0(\kappa),\ R_1, R_2(\kappa)$, which are the stationary probabilities of the server states;

(5) write a function

$$G_{02}(z) = \left(\frac{\mu_1 - \lambda_1}{\mu_1 - \lambda_1 z}\right)^{\frac{\lambda_1}{\sigma_1}} \frac{\mu_1 - \lambda_1}{\mu_1};$$

(6) count the derivatives z from the functions $G_k(z,x), k = 0,1,2$ and $G_{02}(z,x)$:

$$\frac{\partial G_{02}(z)}{\partial z} = G_{02}(z)\frac{\lambda_1}{\sigma_1}\frac{\lambda_1}{\mu_1 - z\lambda_1}; \qquad \frac{\partial G_1(z)}{\partial z} = \frac{\lambda_1 G_1(z) + \lambda_1 \dfrac{\partial G_{02}(z)}{\partial z}}{\mu_1 - z\lambda_1},$$

$$\frac{\partial G_0(z,x)}{\partial z} = \frac{-(\lambda_1+\lambda_2)G_0(z,x) + \mu_1 G_1(z) + \mu_2 G_2(z,x) - xG_0(z,x)}{\sigma_1 z},$$

$$\frac{\partial G_2(z,x)}{\partial z} = \frac{(\lambda_2 + x)G_{02}(z) - (\lambda_1+\lambda_2+\mu_2+x)G_2(z,x)}{\sigma_1 z},$$

$$\frac{\partial^2 G_2(z,x)}{\partial z^2} = \frac{(\lambda_2 + x)G_{02}(z) - (\lambda_1+\lambda_2+\mu_2+x+\sigma_1)\dfrac{\partial G_2(z,x)}{\partial z}}{\sigma_1 z};$$

(7) write $A(z,x)$ by the formula

$$A(z,x) = \frac{1}{z-1}\Big((a(x)+x)G_0(z,x) + z(a(x)-\lambda_2)G_1(z)$$
$$-(\lambda_1 z + \lambda_2 - a(x))G_2(z,x) - \sigma_1 z\frac{\partial G_2(z,x)}{\partial z}\Big),$$

$B(z,x)$ by the formula (27), for the $h_{02}(z,x)$ – (25):

$$h_{02}(z,x) := if\left(z = 1, \frac{1}{\mu_1}A(1,x), h_{02}(z,x)\right);$$

(8) determine $D(z,x)$ (24) and $h_2(1,x)$ (23);

(9) write the diffusion coefficient $b(x)$ (22);

(10) construct $P1(i)$ (32):

$$P1(i) = \frac{1}{b(\sigma_2 i)}\exp\left(\frac{2}{\sigma_2}\int\limits_0^{\sigma_2 i}\frac{a(x)}{b(x)}dx\right);$$

(11) perform normalization and obtain an approximation of the discrete probability distribution of the number of calls on the non-priority orbit

$$P(i) := P1(i)\left(\sum_{i=0}^{N} P1(i)\right)^{-1}.$$

Example 1. Set the parameters of the system:

$$\lambda_1 = 0.3,\quad \lambda_2 = 0.9,\quad \mu_1 = 1,\quad \mu_2 = 2,\quad \sigma_1 = 1.$$

We obtain the probability distribution of the server states:

$$R_0 = 0.25,\quad R_1 = 0.3,\quad R_2 = 0.45.$$

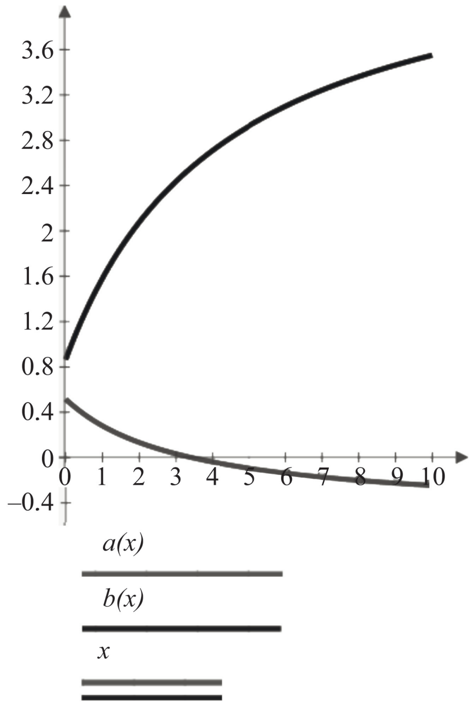

Fig. 2. Transfer coefficient and diffusion coefficient plots.

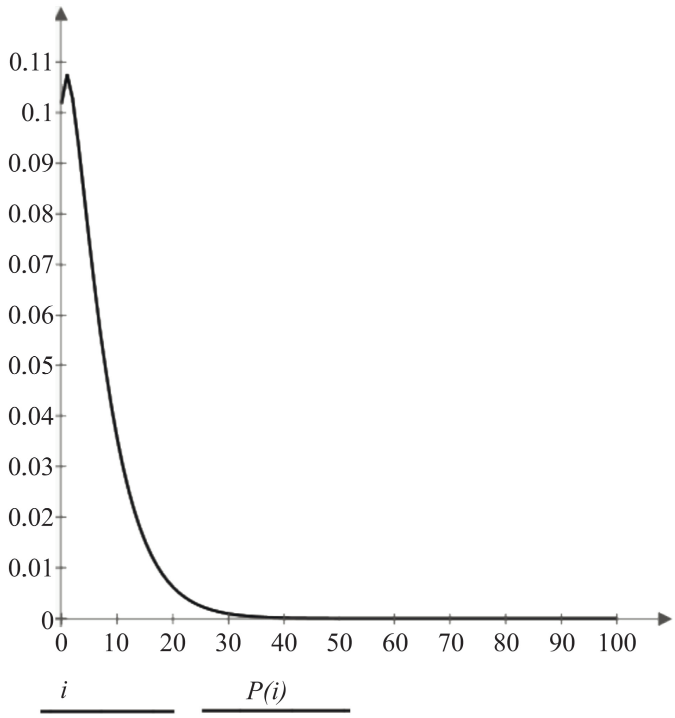

Fig. 3. Diffusion approximation $P(i)$ at $\sigma_2 = 1,\ N = 100$.

Figure 2 shows the transfer coefficient $a(x)$ and diffusion coefficient $b(x)$ dependence on the number of calls in the non-priority orbit. We can conclude that as the number of applications increases, the spread relative to the average increases. Figures 3–5 show diffusion approximation plots of probability distribution of the number of calls on the non-priority orbit.

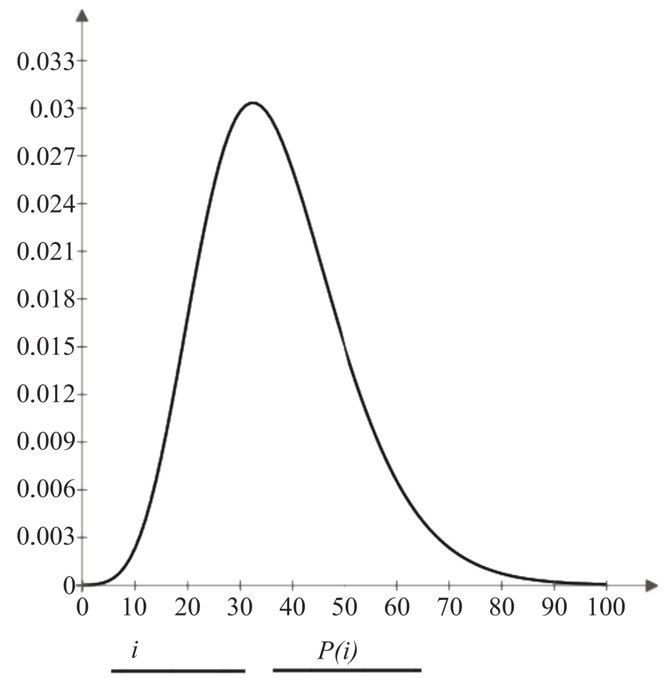

Fig. 4. Diffusion approximation $P(i)$ at $\sigma_2 = 0,1,\ N = 100$.

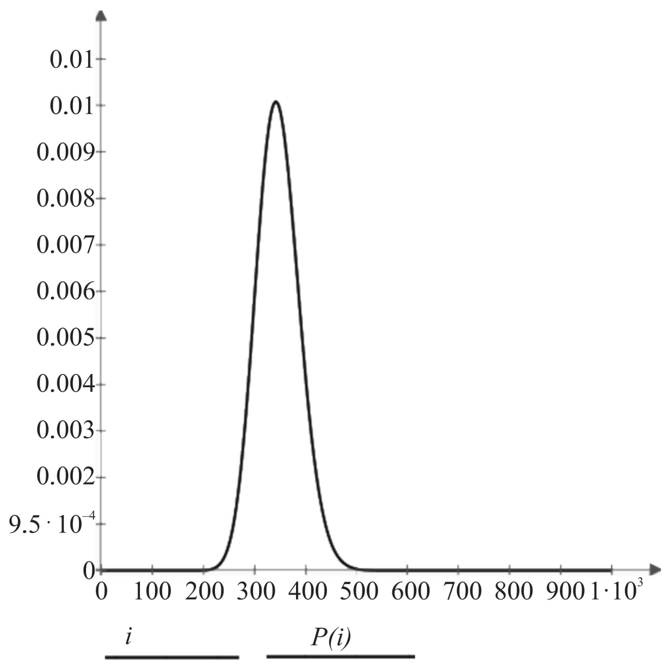

Fig. 5. Diffusion approximation $P(i)$ at $\sigma_2 = 0.01,\ N = 1000$.

Numerical results show the proposed research approach implementing possibility to problems of this type. Also, we can conclude from the distribution graph that for small values of σ_2 the distribution is close to the Gaussian probability distribution. This conclusion can be confirmed by referring to [15], where the system $M^{(2)}|M^{(2)}|1$ is studied by the Gaussian approximation method.

5. CONCLUSION

In this paper, a queuing system with repeated calls and two incoming flows of calls (priority and non-priority stream) is investigated.

For the number of priority calls on the orbit, the derivative function is found in the form of a weighted sum of derivative functions of negative-binomial distributions.

For the number of non-priority requests in an orbit, we obtain a probability distribution, which we call the diffusion approximation.

The found probability distributions allow us to determine all the necessary probabilistictemporal characteristics of the system for the received number of calls. The obtained theoretical results can be used to solve a number of practical problems in which it is necessary to separate requests by priority: cognitive radio, data transmission, where it is necessary to separate by the volume of transmitted data packets. The investigation of problems of a more general type (with arbitrary service time distribution function) is of concern.

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